

# Local Gauge Symmetry and its Breaking as Higgs Couplings in Atomic Quantum Simulation of Dynamical Gauge Fields

Kenichi Kasamatsu<sup>1</sup>, Ikuo Ichinose<sup>2</sup>, and Tetsuo Matsui<sup>1</sup>

<sup>1</sup>*Department of Physics, Kinki University, Higashi-Osaka, Osaka 577-8502, Japan*

<sup>2</sup>*Department of Applied Physics, Nagoya Institute of Technology, Nagoya, 466-8555 Japan*  
(Dated: January 22, 2013)

Recently, the possibility of atomic quantum simulation of dynamical gauge fields was pointed out. In this simulation, a system of cold atoms trapped on each link in an optical lattice is regarded as a simulator of a U(1) lattice gauge model under certain conditions. However, to implement local gauge invariance, a fine tuning of interaction parameters among atoms is necessary. We consider a general cold-atom system with this *fine tuning being relaxed* and investigate the effect of gauge-symmetry breaking. We reveal the fact that a wide variety of cold atoms put on the links of an optical lattice is still to be a faithful quantum simulator for a *lattice U(1) gauge-Higgs model* containing a Higgs field sitting on site. Physical meaning of the Higgs phase in various quantum simulators is also discussed.

PACS numbers: 03.75.Hh, 11.15.Ha, 67.85.Hj, 05.70.Fh, 64.60.De

In the last decade, the idea to use ultracold atoms in an optical lattice (OL) as a simulator for various models in quantum physics seems to become more and more realistic [1, 2]. In particular, one of the interesting possibilities is to simulate lattice gauge theories by putting several kinds of cold atoms on links of an OL in a certain rule [3–11]. Several proposals for pure U(1) lattice gauge theories (LGT) [12] were given in Refs.[3–8] and later the proposal was extended to quantum electrodynamics with dynamical fermionic matter [9, 10] and non-Abelian gauge models [11].

Shortly after the introduction by Wilson [13], LGT has been studied quite extensively mainly in the context of high-energy physics both by analytical methods and by Monte Carlo simulations, and its various properties have been clarified so far. However, the above mentioned approach using cold atoms in an OL provides us with another interesting method for studying LGT. As an example of expected results, the authors of Ref.[6] refer to clarification of dynamics of electric strings in the confinement phase. Furthermore, the quantum simulation is not suffered from the sign problem in contrast to the conventional Monte Carlo simulation of fermions, and it also gives important insight to the real-time evolution of quark-gluon plasma emerging from a heavy-ion collision.

One characteristic point of this cold-atom approach is that the relation to the gauge system is established only under some specific conditions. For example, in Refs.[6–11], one needs fine tuning of a set of interaction parameters; in other words, the *local gauge symmetry is explicitly lost* when these parameters deviate from their optimal values.

The above mentioned point naturally poses us a serious and important question on the stability of gauge symmetry, and potential subtlety of results of experiments of cold atoms as simulators of LGTs, because generally the above conditions are not satisfied exactly or easily violated in actual cold atomic systems. In this Letter, we address this problem semi-quantitatively and exhibit

the allowed range of violation of above conditions, such as the regime of interaction parameters, within which the results can be regarded as its own properties of the LGT.

Explicitly, we shall argue that a U(1) LGT in a four-dimensional (4D) lattice coupled with a Higgs field defined on sites is the target gauge system of atomic quantum simulation by cold atoms. We take the London limit of the Higgs field and ignore its radial degrees of freedom for the reason that will become clear shortly. The Higgs couplings may be nearest-neighbor (NN) ones and/or next-NN ones. These Higgs couplings, as we shall explain, effectively describes the effect of “gauge-symmetry breaking” terms of the original cold atoms in a 3D OL.

Let us start with the path-integral representation of the partition function  $Z$  of the compact U(1) pure LGT, the reference system of the present study:

$$\begin{aligned} Z &= \int [dU] \exp(A), \quad \int [dU] \equiv \prod_{x,\mu} \int_0^{2\pi} \frac{d\theta_{x\mu}}{2\pi}, \\ A &= \frac{c_2}{2} \sum_x \sum_{\mu < \nu} \bar{U}_{x\nu} \bar{U}_{x+\nu,\mu} U_{x+\mu,\nu} U_{x\mu} + \text{c.c.} \\ &= c_2 \sum_x \sum_{\mu < \nu} \cos \theta_{x\mu\nu}, \quad \theta_{x\mu\nu} \equiv \nabla_\mu \theta_{x\nu} - \nabla_\nu \theta_{x\mu}, \\ U_{x\mu} &\equiv \exp(i\theta_{x\mu}), \quad \nabla_\mu f_x \equiv f_{x+\mu} - f_x, \end{aligned} \quad (1)$$

where  $x = (x_1, x_2, x_3, x_4)$  is the site index of the 3+1=4D lattice ( $x_4$  is the imaginary time in the path-integral approach) and  $\mu$  and  $\nu$  ( $= 1, 2, 3, 4$ ) are the direction indices and we use them also as the unit vectors in the  $\mu$ -th and  $\nu$ -th direction. The angle variable  $\theta_{x\mu} \in [0, 2\pi)$  and its exponential  $U_{x\mu}$  are the gauge variable defined on the link  $(x, x+\mu)$  [14]. The bar in  $\bar{U}_{x\mu}$  implies complex conjugate, and  $c_2 (\equiv 1/e^2)$  is the inverse self gauge coupling constant. The product of four  $U_{x\mu}$  is invariant under the following local ( $x$ -dependent) U(1) gauge transformation,

$$U_{x\mu} \rightarrow U'_{x\mu} \equiv V_{x+\mu} U_{x\mu} \bar{V}_x, \quad V_x \equiv \exp(i\Lambda_x), \quad (2)$$

and so are the field strength  $\theta_{x\mu\nu}$  and the action  $A$ .

It is known [15] that the system has a weak first-order phase transition at  $c_2 = c_{2c} \simeq 1.0$ . For  $c_2 < c_{2c}$  the system is in the confinement phase in which the fluctuations of  $\theta_{x\mu}$  is strong. For  $c_2 > c_{2c}$ , the system is in the Coulomb phase in which the fluctuations of  $\theta_{x\mu}$  is small and  $\theta_{x\mu}$  describes almost-free massless particles, which corresponds to the photon in the electromagnetism [14].

To obtain the quantum Hamiltonian  $\hat{H}$  for  $Z$ , let us focus on the space-time plaquette term  $\cos \theta_{xi4}$  in  $Z$  with the spatial direction index  $i(=1, 2, 3)$  and rewrite it as

$$\begin{aligned} \exp(c_2 \cos \theta_{xi4}) &\simeq \sum_{m_{xi} \in \mathbf{Z}} \exp \left[ -\frac{c_2}{2} (\theta_{xi4} - 2\pi m_{xi})^2 \right] \\ &\propto \sum_{E_{xi} \in \mathbf{Z}} \exp \left[ -iE_{xi}(\nabla_i \theta_{x4} - \nabla_4 \theta_{xi}) - \frac{1}{2c_2} E_{xi}^2 \right], \end{aligned} \quad (3)$$

where we used the Villain (periodic Gaussian) approximation in the first line and the Poisson's summation formula in the second line. The term  $iE_{xi} \nabla_4 \theta_{xi} \simeq i d\tau E_{xi} \dot{\theta}_{xi}$  ( $\tau$  is the imaginary time and  $\dot{\theta} \equiv d\theta/d\tau$ ) shows that the integer-valued field  $E_{xi}$  on the spatial link  $(x, x+i)$  is the conjugate momentum of  $\theta_{xi}$ . Thus the corresponding operators at spatial site  $r = (x_1, x_2, x_3)$  satisfy the canonical commutation relation  $[\hat{E}_{ri}, \hat{\theta}_{r'i'}] = -i\delta_{rr'}\delta_{ii'}$ .  $\hat{E}_{ri}$  is the electric field in the electromagnetism but has integer eigenvalues due to the compactness (periodicity) of  $A$  under  $\theta_{x\mu} \rightarrow \theta_{x\mu} + 2\pi$ .

The integration over  $\theta_{x4}$  can be performed as

$$\begin{aligned} G &\equiv \int \prod_x d\theta_{x4} \exp(-i \sum_{x,i} E_{xi} \nabla_i \theta_{x4}) = \prod_x \delta_{Q_x, 0}, \\ Q_x &\equiv \sum_i \nabla_i E_{xi}, \end{aligned} \quad (4)$$

where we used  $\sum_{x,i} E_{xi} \nabla_i \theta_{x4} = -\sum_{x,i} \nabla_i E_{xi} \cdot \theta_{x4}$  which holds for a lattice with periodic boundary condition.

One may check that the quantum Hamiltonian  $\hat{H}$  corresponding to  $Z$  is just the one given by Kogut and Susskind [16],

$$\hat{H} = \frac{1}{2c_2} \sum_{r,i} \hat{E}_{ri}^2 - c_2 \sum_{r,i < j} \cos \hat{\theta}_{rij}. \quad (5)$$

The second term corresponds to the magnetic energy  $(\vec{\nabla} \times \vec{A})^2$  in the continuum [14]. In fact, by inserting the complete sets  $\hat{1}_E = \prod_{r,i} \sum_{E_{ri}} |\{E_{ri}\}\rangle \langle \{E_{ri}\}|$  and  $\hat{1}_\theta = \prod_{r,i} \int d\theta_{ri} |\{\theta_{ri}\}\rangle \langle \{\theta_{ri}\}|$ , in between the short-time Boltzmann factors  $\exp(-\Delta\tau \hat{H})$  ( $\Delta\tau \equiv \beta/N$ ), one may derive the relation,  $Z = \text{Tr } \hat{G} \exp(-\beta \hat{H})$ ,  $\hat{G} \equiv \prod_r \delta_{\hat{Q}_r, 0}$ ,  $\hat{Q}_r \equiv \nabla_i \hat{E}_{ri}$  [17].  $\hat{Q}_r$  is the generator of time-independent gauge transformation and  $\hat{H}$  respects this symmetry as  $[\hat{H}, \hat{Q}_r] = 0$ .  $\hat{Q}_r = 0$  is the Gauss's law to be imposed for physical states as a constraint.

Let us turn to the cold atom studies in Refs.[3–11], and focus on the quantum simulator using Bose-Einstein condensation (BEC) [6] for concreteness. We write the boson

operator on the link as  $\hat{\psi}_{ri} = \sqrt{\hat{\rho}_{ri}} \exp(i\hat{\theta}_{ri})$ , where we use the same letter  $\theta_{ri}$  as  $\theta_{x\mu}$  in Eq.(1) because the former shall be identified as the latter. For a homogeneous distribution of atoms, we set  $\hat{\rho}_{ri} = \rho_0 + \hat{\eta}_{ri}$  and take  $\hat{\eta}_{ri}$  as fluctuation operator. Then the Hamiltonian may take a form as

$$\begin{aligned} \hat{H}_{\text{atom}} &= \frac{1}{2\gamma^2} \sum_r \left( \sum_i \nabla_i \hat{\eta}_{ri} \right)^2 + V_0 \sum_{r,i} \hat{\eta}_{ri}^2 \\ &+ H'_{\text{atom}}(\{\hat{\theta}_{ri}\}) + O(\eta^3), \end{aligned} \quad (6)$$

and  $H'_{\text{atom}}$  is the residual interactions. We use the coherent state  $|\{\psi_{ri}\}\rangle$  and  $\hat{1} = \prod_{r,i} \int d\rho_{ri} d\theta_{ri} \cdot |\{\psi_{ri}\}\rangle \langle \{\psi_{ri}\}|$  to obtain the path-integral expression of  $Z_{\text{atom}} = \text{Tr } \exp(-\beta \hat{H}_{\text{atom}})$  as

$$\begin{aligned} Z_{\text{atom}} &= \int [d\psi] \exp \left[ - \sum_{x,i} \bar{\psi}_{xi} \nabla_4 \psi_{xi} - \Delta\tau \sum_{x4} H_{\text{atom}}(\psi_{xi}) \right] \\ &= \int \prod_{x,i} [d\eta_{xi} d\theta_{xi}] \exp \left[ \sum_{x,i} \left( -i\eta_{xi} \nabla_4 \theta_{xi} - \Delta\tau V_0 \eta_{xi}^2 \right) \right. \\ &\quad \left. - \frac{\Delta\tau}{2\gamma^2} \sum_x \left( \sum_i \nabla_i \eta_{xi} \right)^2 - \Delta\tau \sum_{x4} H'_{\text{atom}}(\{\theta_{xi}\}) \right]. \end{aligned} \quad (7)$$

The first term in the exponent of R.H.S. of Eq.(7) comes from  $\sum_{x4} \bar{\psi}_{xi} \nabla_4 \psi_{xi} \simeq i \sum_{x4} \eta_{xi} \nabla_4 \theta_{xi}$  and shows  $-\hat{\eta}_{ri}$  is the conjugate momentum of  $\hat{\theta}_{ri}$ , so  $\hat{E}_{ri} = -\hat{\eta}_{ri}$ . It implies that the first term of Eq.(6) is  $(2\gamma^2)^{-1} \sum_x \hat{Q}_x^2$ . With  $Q_x = -\sum_i \nabla_i \eta_{xi}$ , we write this factor in Eq.(7) as

$$e^{-\frac{\Delta\tau}{2\gamma^2} Q_x^2} \simeq \int_0^{2\pi} \frac{d\theta_{x4}}{2\pi} e^{\frac{\gamma^2}{\Delta\tau} \cos \theta_{x4} - i\theta_{x4} \sum_i \nabla_i \eta_{xi}}. \quad (8)$$

This Gaussian factor shows that the constraint  $Q_x = 0$  for  $\gamma = 0$  is shifted to a Gaussian distribution with  $Q_x^2 \lesssim \gamma^2/\Delta\tau$  for  $\gamma \neq 0$ ;  $\gamma$  is a parameter to measure the violation of Gauss's law.

In Eqs.(7,8), if one integrates over  $\eta_{xi} \in (-\infty, \infty)$ , one obtains a term  $-(4\Delta\tau V_0)^{-1} (\nabla_4 \theta_{xi} - \nabla_i \theta_{x4})^2$ ; a part of Gaussian Maxwell term. However, this result should be improved to respect the periodicity under  $\theta_{xi} \rightarrow \theta_{xi} + 2\pi$ , because  $\theta_{xi}$  is the phase of the condensate. This Gaussian term is to be replaced, e.g., by a periodic Gaussian form or the corresponding cosine form  $\cos \theta_{xi4}$  like Eq.(3) (This may be achieved by *summing over integer*  $\eta_{xi}$ ).

Then we come to consider the following lattice model  $Z_\gamma$  with compact gauge variables  $\theta_{x\mu}$ ,

$$Z_\gamma = \int [dU] \exp \left( \sum_{x,\mu} c_{1\mu} \cos \theta_{x\mu} + \sum_{x,\mu < \nu} c_{2\mu\nu} \cos \theta_{x\mu\nu} \right), \quad (9)$$

with  $c_{14} = \gamma^2/\Delta\tau$  and  $c_{2i4} \simeq (2\Delta\tau V_0)^{-1}$ . Here, the spatial plaquette  $c_{2ij}$ -terms in Eq.(9) may come from  $H'_{\text{atom}}$  [6] and the  $c_{1i}$ -terms are also added for generality mimicking the LGT [There may be other terms in  $H'_{\text{atom}}$ ; See Eq.(13)].

The model  $Z_\gamma$  is equivalent to *another LGT with exact  $U(1)$  gauge invariance*, i.e.,  $U(1)$  Higgs model with asymmetric NN Higgs couplings  $c_{1\mu}$ . Its partition function  $Z_I$  is defined as

$$Z_I = \int [dU][d\phi] \exp(A_I), \quad \int [d\phi] \equiv \prod_x \int_0^{2\pi} \frac{d\varphi_x}{2\pi},$$

$$A_I = \sum_{x,\mu} c_{1\mu} \cos(\varphi_x + \theta_{x\mu} - \varphi_{x+\mu}) + \sum_{x,\mu<\nu} c_{2\mu\nu} \cos \theta_{x\mu\nu}. \quad (10)$$

$\varphi_x$  is the phase of the Higgs field  $\phi_x$  defined on the site  $x$  with its radial excitation frozen to unity, i.e.,  $\phi_x = \exp(i\varphi_x)$ . So the first term in  $A_I$  is the I-shape hopping term  $c_{1\mu}\bar{\phi}_{x+\mu}U_{x\mu}\phi_x + \text{c.c.}$ .  $A_I$  is gauge invariant under a simultaneous transformation of Eq.(2) and

$$\phi_x \equiv e^{i\varphi_x} \rightarrow \phi'_x = V_x \phi_x \quad (\varphi_x \rightarrow \varphi'_x = \varphi_x + \Lambda_x). \quad (11)$$

In fact,  $Z_\gamma$  is nothing but the gauge fixed version of  $Z_I$  with the so-called unitary gauge  $\varphi_x = 0$ . In short, *the Higgs field  $\phi_x$  represents a fictitious charged matter field to describe the violation of chargeless Gauss's law in the ultra-cold atoms*, where the general Gauss's law with charged field is intact.

The equivalence  $Z_\gamma = Z_I$  is an example of the following general rule: Let us start with a general action  $\tilde{A}(\{U_{x\mu}\})$ , which is composed of  $U(1)$  gauge bit  $U_{x\mu} = \exp(i\theta_{x\mu})$  but may not be gauge-invariant under Eq.(2). Then one may replace  $U_{x\mu} \rightarrow \bar{\phi}_{x+\mu}U_{x\mu}\phi_x$  in  $\tilde{A}(\{U_{x\mu}\})$  to obtain a Higgs action  $\tilde{A}_H(\{U_{x\mu}\}, \{\phi_x\}) \equiv \tilde{A}(\{\bar{\phi}_{x+\mu}U_{x\mu}\phi_x\})$ , which is necessarily gauge-invariant under Eqs.(2) and (11). Then  $\tilde{A}(\{U_{x\mu}\})$  may be viewed as a gauge-fixed version of  $\tilde{A}_H$  to the gauge  $\phi_x = 1$  ( $\varphi_x = 0$ ), and their partition functions  $Z_{\tilde{A}}$  and  $Z_{\tilde{A}_H}$  are equal, i.e.,

$$\int [dU] e^{\tilde{A}(\{U_{x\mu}\})} = \int [dU][d\phi] e^{\tilde{A}(\{\bar{\phi}_{x+\mu}U_{x\mu}\phi_x\})}. \quad (12)$$

This is proved by the change of variable in R.H.S. as  $U_{x\mu} \rightarrow U'_{x\mu} \equiv \bar{\phi}_{x+\mu}U_{x\mu}\phi_x$  before  $\phi_x$  integration and using  $\bar{\phi}_x\phi_x = 1$ ,  $[dU'] = [dU]$  (Haar measure), and  $\int [d\phi] 1 = 1$ .

In Fig. 1, we show the phase diagram of  $Z_I$  in the  $c_2 - c_1$  plane obtained by Monte Carlo simulation for the case  $c_{1\mu} = c_1$  (See Model I there). It shows that the confinement and Coulomb phases, which exist in  $Z$  of Eq. (1), survives only up to the phase boundary  $c_1 = c_{1c}(c_2)$ , and beyond it the system goes into a new phase, the Higgs phase in which both  $\theta_{x\mu}$  and  $\varphi_x$  are stable [18]. The expectation that the cold atoms may simulate the pure gauge theory is assured in qualitative and global sense as long as both systems are in the same phases. This happens up to the parameters corresponding to  $c_1 < c_{1c}(c_2)$ .

The three phases can be characterized by the potential energy  $V(r)$  between two static charges with opposite signs and separated by the distance  $r$  as  $V(r) \propto 1/r$  (Coulomb),  $\exp(-mr)/r$  (Higgs),  $r$  (confinement).

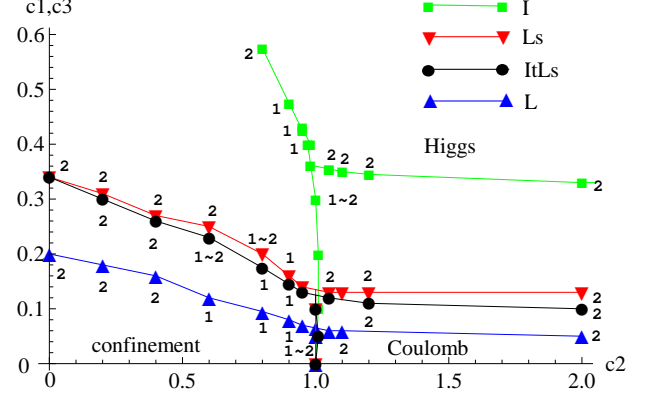


FIG. 1: (Color online) Phase diagram of four models (15) in the  $c_2 - c_{1,3}$  plane for the size  $16^4$ . The vertical axis is  $c_1$  for Model I,  $c_3$  for Models L and Ls,  $c_1 = c_3$  for Model ItLs. All the four models have the three phases. The number (1, 2) at each critical point indicates its order of transition. The Higgs-confinement lines of Model I terminates at  $c_2 \sim 0.7$ .

One may distinguish each phase in the experiments of cold atoms by measuring atomic density (See Fig.2).

The above relation (12),  $Z_\gamma = Z_I$ , leads us to a very interesting interpretation that the cold atomic systems proposed in Ref.[6] and the other related models [7, 8] have a potentiality to be a simulator of wider range of field theories. Namely, the system of cold atoms with a general set of values of parameters can be regarded as a

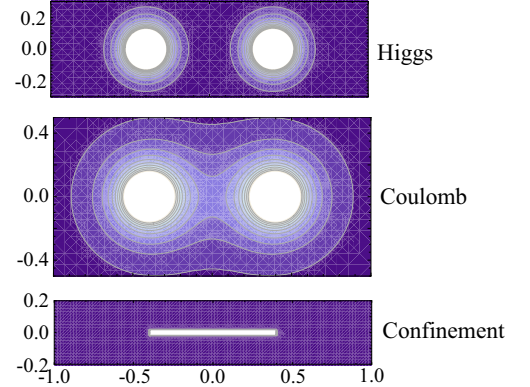


FIG. 2: (Color online) Contour plot of deviation of typical atomic density  $\Delta\rho_r \equiv (\sum_i \eta_{ri}^2/3)^{1/2}$  in the  $x_1 - x_2$  plane at  $x_3 = 0$  with the external sources of atoms  $\Delta\rho_{\text{ext}} = \pm\rho_1$  put on the links emanating from  $r = r_\pm = (\pm 0.4, 0, 0)$ . The white regions have  $\Delta\rho_r$  greater than certain value and the darker regions have the lower  $\Delta\rho_r$ . The atomic density on the link  $(r, r+i)$  is given by  $\rho_{ri} = \rho_0 + \eta_{xi}$ , and the deviation  $\eta_{xi}$  is calculated by using the electric field  $E_{ri} (= -\eta_{ri})$  with a pair of external sources  $q = \pm 1$  at  $r = r_\pm$ . In the Higgs phase,  $\Delta\rho_r$  decreases rapidly away from the sources. In the confinement phase, the deviation propagates from one source to the other along a one-dimensional string (electric flux).

simulator of the U(1) LGT with the Higgs couplings.

In all the approaches to quantum simulator of the dynamical gauge field presented so far [6–11], Hamiltonian of the original system has the form like  $\hat{H}_{\text{atom}} = (2\gamma^2)^{-1} \sum_r \hat{Q}_r^2 + V_0 \sum_{r,i} \hat{E}_{ri}^2 + g' \sum_{r,i,j} (\hat{U}_{ri}^\dagger \hat{U}_{r+i,j} + \text{H.c.}) + (\text{matter})$ . Then, by considering the limit  $\gamma \rightarrow 0$  and making the second-order perturbation theory for small  $g'$ , one obtains a gauge-invariant effective Hamiltonian with the plaquette  $c_2$  term as in Eqs.(5,9)[6–8, 10]. However, for  $\gamma \neq 0$ , the original L-shape terms  $\hat{U}_{ri}^\dagger \hat{U}_{r+i,j}$  ( $\in H'_{\text{atom}}$ ) remain in the effective Hamiltonian as a first-order contribution. For large  $g'$ , they shall certainly play an important role. These L-shape interactions correspond to the following terms in the effective action,

$$A_\delta = \sum_{x,\mu < \nu} c_{3\mu\nu} \left[ \cos(\theta_{x\mu} - \theta_{x\nu}) + \cos(\theta_{x\mu} + \theta_{x+\mu,\nu}) + \cos(\theta_{x+\mu,\nu} - \theta_{x+\nu,\mu}) + \cos(\theta_{x\nu} + \theta_{x+\nu,\mu}) \right]. \quad (13)$$

Here,  $c_{3\mu\nu} = c_3 \xi_{\mu\nu}$  is the coupling between the NN atoms on links and  $\xi_{i4} = 0$ . Then, by the same token, this term is viewed as a gauge fixed version of the gauge invariant term  $A_L(\{U_{x\mu}\}, \{\phi_x\})$  obtained by replacing  $\theta_{x\mu} \rightarrow \varphi_x + \theta_{x\mu} - \varphi_{x+\mu}$  in  $A_\delta$ .

The above consideration for cold atoms motivates us to introduce the following general Higgs couplings:

$$\begin{aligned} Z_{\text{IL}} &= \int [d\phi][dU] \exp A_{\text{IL}}(\{U_{x\mu}\}, \{\phi_x\}), \\ A_{\text{IL}} &= A_{\text{I}} + A_{\text{L}}, \\ A_{\text{L}} &= \sum_{x,\mu < \nu} c_{3\mu\nu} \left[ \cos(\varphi_{x+\nu} + \theta_{x\mu} - \theta_{x\nu} - \varphi_{x+\mu}) \right. \\ &\quad + \cos(\varphi_x + \theta_{x\mu} + \theta_{x+\mu,\nu} - \varphi_{x+\mu+\nu}) \\ &\quad + \cos(\varphi_{x+\mu} + \theta_{x+\mu,\nu} - \theta_{x+\nu,\mu} + \varphi_{x+\nu}) \\ &\quad \left. + \cos(\varphi_x + \theta_{x\nu} + \theta_{x+\nu,\mu} - \varphi_{x+\nu+\mu}) \right]. \quad (14) \end{aligned}$$

Each term in  $A_{\text{L}}$  may be written as  $\bar{\phi}_{x+\mu+\nu} U_{x+\mu,\nu} U_{x\mu} \phi_x$ , etc. To study the phase structure of  $Z_{\text{IL}}$ , we consider the following cases for definiteness;

$$\begin{aligned} c_{1\mu} &= c_1 \xi_\mu, \quad c_{2\mu\nu} = c_2, \quad c_{3\mu\nu} = c_3 \xi_{\mu\nu}, \\ \text{Model } \xi_i \quad \xi_4 \quad \xi_{ij} \quad \xi_{i4} \\ \text{I} &\quad 1 \quad 1 \quad 0 \quad 0 \\ \text{L} &\quad 0 \quad 0 \quad 1 \quad 1 \\ \text{Ls} &\quad 0 \quad 0 \quad 1 \quad 0 \\ \text{ItLs} &\quad 0 \quad 1 \quad 1 \quad 0 \end{aligned} \quad (15)$$

Model ItLs (t is for time and s is for space) is the nearest to the systems studied in Refs.[6, 7] [19].

Figure 1 is the phase diagram of four Models in Eq.(15) in the  $c_2 - c_{1,3}$  plane. These four phase diagrams have similar structure. There are always three phases; Higgs, Coulomb, and confinement phases in the order of increasing size of fluctuations of gauge field  $\theta_{x\mu}$ . The order

of phase transitions is weak-first order or second-order in the confinement-Coulomb transition, second-order in the Coulomb-Higgs transition, while it changes in the confinement-Higgs transition as first-order, second-order, and cross over (for Model I) [20], as  $c_2$  decreases from the tricritical point at which the three phases merge.

For sufficiently large  $c_2$ ,  $\theta_{x\mu}$  is almost frozen up to gauge transformation, and the system reduces to the XY model [18]. Then, for Model I  $\phi_x$  has the NN interaction,  $c_{1\mu} \bar{\phi}_{x+\mu} \phi_x$ , while for Models L and Ls the interaction is next NN one,  $c_{3\mu\nu} \bar{\phi}_{x+\mu+\nu} \phi_x$ . These XY models exhibit second-order transition both in 3D(Model Ls) and in 4D(Models I, L) between ferromagnetic and paramagnetic phases, which corresponds to the Higgs-Coulomb transition in Fig.1. The difference of number of interaction bonds per site, 4, 12, 24 for Models I, Ls, L, makes the region of the ordered phase (Higgs phase) larger in this order [21].

It is quite instructive to clarify physical meaning of the Higgs phase of the gauge system realized in the atomic quantum simulators. In the simulator using bosons [6], the Higgs phase of the effective gauge system is nothing but the BEC state as the phase of the bosons (i.e., the gauge boson) is stabilized coherently. Therefore, the Higgs-confinement transition corresponds to the BEC transition. On the other hand, in the quantum simulator using the Schwinger boson  $z_r^\sigma$  for the gauge field like  $\hat{U}_{ri} = \hat{z}_{r+i}^\dagger \hat{z}_r^\sigma$  ( $\sigma = 1$  or  $2$ ) [9, 11], the Higgs phase corresponds to the state in which quantum state at each link ( $r, r+i$ ) is given by a coherent superposition of the particle-number states like  $|0\rangle_r |1\rangle_{r+i} + |1\rangle_r |0\rangle_{r+i}$ . In the double-well potential, this state is realized naturally and then the Higgs phase of the gauge system appears easily.

This way of introducing U(1) variables [9, 11] reminds us an approach of starting with the antiferromagnet with  $s = 1/2$  quantum spin at each site and obtaining the  $\text{CP}^1 + \text{U}(1)$  LGT [22], which has a two-component complex ( $\text{CP}^1$ ) variable at each site describing spins and an *auxiliary but dynamical U(1) gauge variables* on each link. Although the  $\text{CP}^1 + \text{U}(1)$  model and the present U(1) Higgs model are different each other, their global phase structures are quite similar (See Fig.1 of Ref.[22]).

In summary, Fig.1 predicts global phase structures of the effective system (14) of cold atoms trapped on the links of an OL that are studied in Refs.[6, 7], and other similar systems. From the discussion presented in Refs.[6, 7] and the relation (12) between the U(1) gauge invariant Higgs theory and its gauge fixed theory ( $\varphi_x = 0$ ), it may be rather universal that many systems of multiplet (“*quantum spins*”) of cold atoms put on links have their U(1) Higgs LGT counterparts. Such equivalence between cold atoms and U(1) Higgs model may be called a “*quantum spin-Higgs gauge correspondence*”. In the next stage, more explicit and individual relation between a cold atom system and its corresponding LGT (14) may be of interest, which requires expressions of  $c_{1\mu}$  etc. in terms of parameters of the cold atom system.

- 
- [1] M. Lewenstein, A. Sanpera, and V. Ahufinger, *Ultracold Atoms in Optical Lattices: Simulating Quantum Many-body Systems* (Oxford University Press, 2012).
- [2] I. Bloch, J. Dalibard, and S. Nascimbene, Nat. Phys. **8**, 267 (2012).
- [3] H. P. Büchler, M. Hermele, S. D. Huber, M. P. A. Fisher, and P. Zoller, Phys. Rev. Lett. **95**, 040402 (2005).
- [4] S. Tewari, V. W. Scarola, T. Senthil, and S. Das Sarma, Phys. Rev. Lett. **97**, 200401 (2006).
- [5] H. Weimer, M. Müller, I. Lesanovsky, P. Zoller, and H. P. Büchler, Nat. Phys. **6**, 382 (2010).
- [6] E. Zohar and B. Reznik, Phys. Rev. Lett. **107**, 275301 (2011).
- [7] E. Zohar, J. I. Cirac, and B. Reznich, Phys. Rev. Lett. **109**, 125302 (2012).
- [8] L. Tagliacozzo, A. Celi, A. Zamora, and M. Lewenstein, arXiv:1205.0496.
- [9] D. Banerjee, M. Dalmonte, M. Müller, E. Rico, P. Stebler, U.-J. Wiese, and P. Zoller, Phys. Rev. Lett. **109**, 175302 (2012).
- [10] E. Zohar, J. I. Cirac, and B. Reznich, arXiv:1208.4299.
- [11] E. Zohar, J. I. Cirac, and B. Reznich, arXiv:1211.2241; D. Banerjee, M. Bögli, M. Dalmonte, E. Rico, P. Stebler, U.-J. Wiese, and P. Zoller, arXiv:1211.2242; L. Tagliacozzo, A. Celi, P. Orland, and M. Lewenstein, arXiv:1211.2704.
- [12] The word “pure” implies that the system contains only gauge fields and no other fields such as quarks, etc.
- [13] K. Wilson, Phys. Rev. D **10**, 2445 (1974); J. B. Kogut, Rev. Mod. Phys. **51**, 659 (1979).
- [14]  $\theta_{x\mu}$  is related to the vector potential  $A_\mu(x)$  in the continuum space-time as  $\theta_{x\mu} = aeA_\mu(x)$  where  $a$  is the lattice spacing. In the formal continuum limit  $a \rightarrow 0$ , the action is reduced to  $A \rightarrow -(1/4) \int d^4x F_{\mu\nu}(x) F_{\mu\nu}(x)$  with  $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$  as it should be.  $Z$  can be defined without a gauge fixing due to the compactness  $\int [dU] 1 = 1$  in contrast with  $\int_{-\infty}^{\infty} dA_\mu(x) = \infty$ .
- [15] See, e.g., E. Sañchez-Velasco, Phys. Rev. E **54**, 5819 (1996), and references cited therein.
- [16] J. Kogut and L. Susskind, Phys. Rev. D **11**, 395 (1975).
- [17] To respect  $\hat{G}$  in  $Z$ , one needs to insert  $\prod_r \delta_{Q(r,x_4),0}$  at least only once at any  $x_4$  in the path-integral due to  $[\hat{H}, \hat{G}] = 0$ . In other words, one may insert it at every  $x_4$  as done in Eq.(4) due to the equalities,  $\hat{G}^2 = \hat{G}$ ,  $\hat{G} \exp(-\beta \hat{H}) = \hat{G} [\exp(-\beta \hat{H}/N) \hat{G}]^N$ .
- [18] This is consistent with the known result for the XY spin model with the action  $A_{XY} = c_1 \sum_{x,\mu} \cos(\varphi_{x+\mu} - \varphi_x)$ , which exhibits a second-order transition for  $D = 3, 4$ . In fact, if we take  $c_2 \gtrsim 1.0$  in  $Z_1$ ,  $\theta_{x\mu}$  becomes stable with irrelevantly small fluctuations as in the Coulomb phase of  $Z$  of (1), and  $A_I$  reduces to  $A_{XY}$  by putting, e.g.,  $\theta_{x\mu} \simeq 0$ .
- [19] Strictly speaking, the original systems in Ref. [6, 7] correspond to  $c_{2ij} = 0, c_{2i4} \neq 0, c_{3ij} \neq 0, c_{3i4} = 0$ , whereas the effective systems derived by perturbative expansion and restriction of the Hilbert space to the gauge-invariant subspace have  $c_{2ij} \neq 0, c_{2i4} \neq 0, c_{3ij} = c_{3i4} = 0$ .
- [20] For Model I, the second-order transition disappears at certain  $c_2 (> 0)$  and changes to a crossover, because, at  $c_2 = 0$ , the integration over  $\theta_{x\mu}$  is factorized link by link, giving rise to an analytic partition function of  $c_1$ .
- [21] For cold atoms in the 2D OL [6], its effective model is the 2+1=3D U(1) LGT with NN and/or next NN Higgs couplings. The 3D pure U(1) gauge theory is always in the confinement phase ( $0 \leq c_2 < \infty$ ) and the Coulomb phase is impossible[A. M. Polyakov, Phys. Lett. **B59**, 82 (1975)]. Therefore, mapping to the 3D XY model by setting  $U_{x\mu}$  constant [18] does not hold, and the 3D Model I ( $c_1 \neq 0, c_3 = 0$ ) also lives only in the confinement phase. For the 3D Model L,  $U_{x\mu}$  itself works as an XY spin and their NN interaction exhibits 3D XY (confinement-Higgs) transition at  $c_3 = c_{3c} > 0$  with all  $c_2$ .
- [22] K. Sawamura, T. Hiramatsu, K. Ozaki, I. Ichinose, and T. Matsui, Phys. Rev. B **77**, 224404 (2008).